A HYBRID MODEL FOR DESIGNING ATTRIBUTES SAMPLING PLANS

A Thesis
Submitted to The Department of Industrial Engineering
And the Institute of Engineering and Science
of Bilkent University
In Partial Fulfillment of the Requirements
For the Degree of
Master of Science

By
M. BAŞAR ŞAHİN

September, 1994

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I certify that I have read this thesis and that in my opinion it is fully adequate, in scope and in quality, as a thesis for the degree of Master of Science.

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ABSTRACT

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M. BAŞAR SANİN
M.S. in Industrial Engineering
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September 1994

In single sampling plans by attributes, statistical and economical considerations have traditionally been discussed separately. An approach taking into account both considerations simultaneously would be more useful in terms of quality assurance. The suggested model involves minimization of the expected total cost comprising the inspection cost, the annoyance cost of rejecting a lot and the cost of outgoing defective items, while the producer's risk and consumer's risk are not allowed to be greater than specified values. To find the optimal sample size and the optimal acceptance number, a two stage solution method is proposed. The accuracy and the efficiency of the solution procedure are tested on randomly generated problems, by comparing the solutions obtained by the proposed method to those obtained by enumeration. Sensitivity of the model is discussed by analyzing the effects of parameters on the optimal sampling plans.

Keywords: Quality Control, Acceptance Sampling Plans, Attributes Sampling.

ÖZET

ÖZNİTELİK ÖRNEKLEME PLANLARI TASARIMI

İÇİN KARMA BİR MODEL

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Geleneksel olarak, tek aşamalı öznitelik örnekleme planlarında istatistiksel ve ekonomik yaklaşımlar ayrı olarak düşünülmüştür. Kalite güvencesi açısından bu yaklaşımların birleştirilmesi yararlı olacaktır. Önerilen model, inceleme, öbek reddetme ve kusurlu çıktı öğelerin maliyetini göz önünde bulunduran beklenen toplam maliyetin en küçüklenmesini amaçlarken, üreticinin ve tüketicinin risklerinin de belirli değerlerden fazla olmasına izin vermemektedir. En iyi örnekleme büyüklüğü ve en iyi kabul sayısının bulunabilmesi için iki aşamalı bir çözüm yöntemi önerilmiştir. Önerilen yöntem ve birerleme ile çözülen rassal problemlerin sonuçları karşılaştırılarak, yöntemin doğruluğu ve verimliliği test edilmiştir. Değişik parametrelerin sonuçlar üzerindeki etkisi çözümlenerek, modelin duyarlılığı incelenmiştir.

Anahtar Kelimeler: Kalite Denetimi, Kabul Örnekleme Planı, Öznitelik Örneklemesi

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In my opinion, this study was reached an end with love. So I can dedicate it to love.

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1. INTRODUCTION

Inspection of incoming materials, semi-finished products and end-products is a necessary action in manufacturing and is an important part of quality assurance. At every stage of the manufacturing process, materials have to adhere to certain standards. When inspection is carried on for the purpose of accepting or rejecting a product depending on its conformance to specifications, the inspection procedure used is an acceptance sampling. Montgomery [13] defines a typical application of acceptance sampling as follows:

> A company receives a shipment of product from a vendor. This product is often a component or raw material used in the company's manufacturing process. A sample is taken from the lot and some quality characteristic of the units in the sample is inspected. On the basis of the information in this sample, a decision is made regarding lot disposition. Usually, this decision is either to accept or to reject the lot.

Basically, there are three actions to decide on accepting or rejecting a lot. One is to accept the lot with no inspection, the other one is to screen the whole lot (100% inspection), and the last one is to use an acceptance sampling plan. An acceptance sampling plan only prescribes conditions to accept or reject a lot inspected; it is rather an indirect form of quality control.

There are always two parties in this procedure: one is the producer (the supplying party) and the other is the consumer (the buying party) who decides whether the lot is acceptable or not [5]. Accepting a lot which is worse than a specified quality level of consumer is defined as the consumer's risk, while the producer's risk is to reject a lot which is good enough according to the stated quality level of the producer. It is desirable to set up a sampling plan taking into account both the producer's and the consumer's interests. This is beneficial for both parties since their interests are not mutually exclusive.

Statistically, a good acceptance sampling plan considers the risk of producer and the risk of consumer together. Besides the statistical approaches, it is possible to express an acceptance sampling plan in economic terms. In an acceptance sampling plan, for each lot there is an inspection cost and there is a cost for each defective item in an accepted lot. There is a choice of no inspection which corresponds to accepting the lot even though the majority of the items in the lot may be defective. Then there will be no cost of inspection but high cost of defective items. There is also a choice of 100 % inspection, with high inspection cost. In this case, no defective item can go on to the consumer and there is no cost of defective items, assuming no inspection error. Lots that are accepted are sent for further processing (although lot may contain defectives), while rejected lots are reworked or scrapped at some cost.

Schilling [16] relates producer's and consumer's statistical and economical interests as follows: If good lots are rejected, then the producer will lose good products (producer's risk) and the consumer will have higher cost. If bad lots are accepted, then the producer will suffer from customer dissatisfaction and consumer will pay for bad product (consumer's risk).

Traditionally, research on acceptance sampling plans have focused either on statistical or on economical concepts, but rarely on both. A literature review on statistical and economical aspects of single sampling plans by attributes is provided in the next chapter. In Chapter 3, a model taking into account both economical and statistical aspects of an acceptance sampling plan is constructed, its properties are discussed and a solution method is proposed. Chapter 4 gives test results on the accuracy and the efficiency of the proposed solution method, and Chapter 5 discusses sensitivity of the model with respect to problem parameters. Concluding remarks and suggestions for future research are presented in Chapter 6.

2. LITERATURE REVIEW

In practice, acceptance sampling schemes have largely been based on statistical criteria. Schilling [18], provides a background on acceptance quality control prior to 1950's. He mentions that statistical science of acceptance sampling can be traced back to 1920's. The first control charts, the terminology of acceptance sampling, the lot tolerance percent defective (LTPD) sampling tables, the average outgoing quality level (AOQL) sampling tables were developed at those times. In 1940's, sampling inspection tables of Dodge-Romig that provide plans based on fixed consumer risk were published. Advances in variables and attributes sampling and sequential analysis as well as developments in process quality control and industrial and applied statistics lead to issues of Military Standards in 1950's. Military Standards (MIL-STD-105D) is a collection of sampling schemes and is a complete acceptance sampling system [6, 9, 10, 14, 18]. In these tables, acceptable quality limit (AQL) is used as the indexing parameter. It is possible to select different sampling plans depending on the importance of the product, the type of defect, and the quality history. It is possible to designate different AQL's for different types of defects. For a specified AQL and inspection level, and a given lot size, the standard provides a normal sampling plan that is to be used as long as the producer submits lots at a quality level at least as good as AQL. Moreover, it provides a procedure to switch to tightened inspection if the producer's quality deteriorates, and to reduced inspection if submitted quality stays the same or gets better.

Besides the Military Standards, there are a number of procedures used to find the single sampling plans. As noted by Alekseev, Podsevalov and Reinov [1], most of the traditional statistical methods used in sampling plans were developed prior to 1960's. For given values of consumer's risk, producer's risk, AQL and LTPD, related tables are utilized to determine the sampling plans. The tables of Poisson Unity Values constructed by Cameron, and Schilling and Johnson are available in [18]. In these tables, sampling plans are derived for a given operating ratio (ratio of LTPD to AQL) from tabulated acceptance number, consumer's risk and producer's risk values. Poisson approximation to binomial distribution is used in these tables. An example in which Poisson approximation is used, is also presented by Alekseev, Podsevalov and Reinov [1] in a discussion of sampling acceptance control.

Another procedure is Larson's binomial nomograph. Larson's nomograph derives an exact plan for given consumer's risk, producer's risk, AQL and LTPD, and it can also be used to evaluate the operating characteristic curve of a sampling plan [6, 16]. f-binomial approximation of Ladany that provides a method for adopting Larson's nomograph is discussed also in [18]. f-binomial approximation to the hypergeometric distribution is applied to derive plans for finite lots of size N in Ladany's work.

Lastly, the theory of constructing Thorndyke Chart is explained in [16]. Thorndyke Chart provides a procedure to determine a sampling plan based on Poisson distribution for values not available in unity tables.

Other extensively used acceptance sampling schemes are Dodge-Romig Tables. As discussed in [6, 9, 10, 14, 18], two types of sampling plans are given in these tables. The first type of sampling plans offers LTPD protection and the second type offers plans providing a specified AOQL. The design in Dodge-Romig AOQL plans is such that the average total inspection for a given AOQL and a specified process average is minimized. Likewise, average total inspection is minimized in LTPD plans. One restriction on usage of Dodge-Romig plans is that they apply to quality system in which rejected lots are subject to 100 % inspection. Although selection of a Dodge-Romig plan depends on the knowledge of vendor's process average fallout, there are ways to obtain an estimate of that process average.

Kouikoglou [12] presents a solution to the problem of determining a single sampling plan, using Larson's nomograph of the binomial distribution. The author considers constraints on Type I and Type II errors (producer's and consumer's risks, respectively) and shows that there is a feasible region of plans in the nomograph. The author gives numerical examples and comments on the effects of approximation errors.

In recent years increased attention has been devoted to acceptance sampling plans based on economical and/or Bayesian considerations.

Brown and Rutemiller [3] propose tables for determining expected cost per unit under MIL-STD-105D single sampling schemes which are not based upon cost concepts, in fact. They provide the expected fraction of lots rejected, the expected sample size per lot, and the expected number of lots to be processed for several single sampling plans and various quality levels when a lot is subjected to normal, reduced or tightened inspection. They give equations to calculate the long term cost of sampling inspection using these tabulated expected values and relevant cost parameters.

Ercan, Hassan, and Taulananda [7] have investigated a single-stage manufacturing system in which one kind of material procured is first inspected and then processed and finally the finished part is again inspected. They formulate the relationships among average incoming quality level (AIQL), AOQL and process quality level. A model for an optimal single sampling plan is developed and a plan is obtained using discrete optimization for the total expected loss function subject to AIQL and AOQL equality constraints. The total expected loss function is written in terms of loss due to repair and rework / replacement, opportunity loss and loss due to incoming and outgoing inspection costs. In this paper, minimum cost single sampling plans are obtained analytically for the outgoing and incoming inspections where inspections are by attributes.

Moskowitz and Berry [15], have presented a Bayesian algorithm providing a generalized procedure for determining the minimum cost sample size and acceptance number (that may yield no inspection or 100% inspection) for single sample attribute acceptance plans. The algorithm finds the optimal solution by computations over a decision tree. The algorithm provides a general method of determining optimal acceptance plan when the number of defective items in a lot has a discrete distribution and the sampling cost is either a linear or strictly convex function of sample size.

In a similar setting to [15], Tagaras and Lee [19] use modified Beta distribution for the lot fraction defective, and study the properties of optimal Bayesian single sampling plans. Same authors have also investigated a lot by lot production system with two stations in series [13]. They assume a single sampling plan is used at each inspection station, 100 % inspection is applied to a rejected lot and average lot fraction defective for each stage is a random variable. Based on unit inspection cost, unit rework cost and unit cost of outgoing defective items, the expected total cost function of the system is developed. A heuristic algorithm is proposed to determine the optimal sampling plan at each station when average lot fraction defectives have modified Beta distribution.

Bai and Hong [2], consider two different sampling procedures (a fixed size sampling plan with several levels of acceptance numbers and an inverse sampling plan with several levels of sample sizes) to grade the quality of products that can be sold to several markets. A linear profit model is developed considering a product can be sold to a number of different markets. When an accepted lot is sold at a market, a nondefective item yields a certain profit and a defective item yields a certain loss. The sampling plans have costs of inspection and replacement costs. In a fixed size sampling plan, the idea is to find the optimal acceptance number for each market when the expected profit function is the sum of all profit functions of markets. In an inverse sampling plan, the expected profit function is again the sum of all profit functions of markets when the sample size for each market is a decision variable. For both sampling procedures, methods of finding optimal sampling plans are presented in the article.

In an attempt to combine probabilistic aspects and economics of sampling plans, Evans and Alexander [8] present an approach to Bayesian sampling plans that consider two criteria (minimization of cost and minimization of outgoing fraction defective in a lot). Based on multiobjective decision analysis, the authors discuss a methodology that involves the formulation of joint distribution of cost and quality and utility function of the decision maker. Selection of an optimal sampling plan is illustrated via maximization of utility.

Alekseev, Podsevalov and Reinov [1] summarize J. Enell's method to find economic sampling plans. In this method, a critical value for the proportion of defectives in a lot is determined by equating loss associated by accepting one defective unit to the cost of detecting and correcting a defective unit in a rejected lot. Given a lot size, operating characteristic (OC) curves of several sampling plans are constructed and the sampling plan with the curve yielding closest equal risk value to the critical proportion of defectives is selected.

Case and Chen [4] discuss adaptations of an economically based, Bayesian attributes acceptance sampling model. Using different cost terms (for sampling inspection, lot acceptance, lot rejection), they compute total expected cost of acceptance and total expected cost of rejection for known discrete prior distributions of number of defectives in a lot. They state that exact optimization for single sampling consists of determining an optimum acceptance number (the largest value of number of defectives in a sample for which acceptance cost is less than or equal to rejection cost) associated with each sample size and finding minimum cost sampling plan by a search over the sample size. They also present a double sampling model and compare results of single and double sampling. Finally, they comment on actual implementation experiences in industry.

Dinçer [5], presents three economic models of different post-rejection policies. He develops expected cost functions for a single sampling plan in which rejection is followed by 100 % inspection, rejection is followed by a quality guarantee and rejection is followed by resubmission. He analyses these functions for the case of fraction defectives known with certainty and discusses models for the case where the distribution of fraction defectives is known. He also suggests a hybrid model to combine statistical and economical features of sampling plans.

In this study, single sampling plans by attributes are addressed. As mentioned before, a mathematical programming model taking into account both statistical and economical considerations of a sampling system is developed. Average process fraction defective, cost parameters, lot size, LTPD and AQL of the system are assumed to be known and constant. While minimizing the expected cost of inspection, cost of outgoing defectives and cost of quality guarantee, the sampling plan not exceeding certain risk levels for the producer and the consumer is to be determined. It is observed that, due to the nature of the probabilistic functions, the model is hard to solve. A two-stage solution method making use of piecewise linearization of OC curves is proposed to find the optimal solution of the problem, and the solutions of a number of randomly generated problems obtained by this method are compared to the solutions obtained by enumeration. Finally, sensitivity of minimum cost sampling plans have been analyzed with respect to model parameters.

3. HYBRID MODEL

In this chapter, a new approach in designing sampling plans is introduced. The main idea of this model is to utilize the advantages of two main considerations existing in the literature, to overcome their disadvantages. The disadvantage of these two approaches is to consider sampling systems in one direction. Economical approaches try to minimize cost of the system only and ignore the risks that the parties involve. On the other hand, statistical procedures have little or no control over the cost of the procedure. Therefore, a model considering both of them can overcome such unknown and undesirable effects. The model presented in this chapter has first been mentioned by Dinçer [5]. Since the model is complicated to analyze and solve, a solution method is also proposed in this section.

3.1 The Model

The notation used in the suggested model is given below:

model parameters:

α_0 : maximum allowable producer's risk,

β_0 : maximum allowable consumer's risk,

LTPD: acceptable quality limit for consumer,

AQL : acceptable quality limit for producer,

p : average process fraction defective,

N : lot size,

C_i : cost of inspection per unit,

C_d : cost of an outgoing defective unit,

C_A : cost of quality guarantee (cost of rejection, cost of annoyance),

decision variables:

n : sample size,

c : acceptance number,

other notations:

TC : total cost of the sampling plan,

E(TC): expected total cost of the sampling plan,

F(.) : probability of acceptance for a given fraction defective,

α : producer's risk,

β : consumer's risk,

d : random variable denoting the number of defectives in a sample,

and the following relations hold

$$F(AQL) = 1 - \alpha = \sum_{r=0}^{c} \binom{n}{r} (AQL)^r (1 - AQL)^{n-r}$$
$$\cong \sum_{r=0}^{c} \frac{(n\,AQL)^r e^{-n\,AQL}}{r!}, \qquad (3.1).$$

F(AQL) gives the probability of acceptance of lots which contain 100 AQL % defective. The producer desires to keep the acceptance of lots that have a fraction defective of AQL or better as high as possible, does not want to reject lots of high quality (that is, the Type I Error is kept at minimum). On the other hand,

$$F(LTPD) = \beta = \sum_{r=0}^{c} \binom{n}{r} (LTPD)^r (1 - LTPD)^{n-r}$$
$$\cong \sum_{r=0}^{c} \frac{(n\,LTPD)^r e^{-n\,LTPD}}{r!}. \qquad (3.2).$$

Similarly, the consumer desires not to have lots of poor quality, and does not want lots of poor quality to be accepted, hence the acceptance probability of lots containing high number of defectives is kept at minimum (Type II Error). LTPD defines this level of fraction defective that the consumer can tolerate.

The mechanism of a sampling plan can be defined as follows:

A sample of size n is drawn from a lot of size N. If the number of defective items found in the sample is less than or equal to a number c, (namely, acceptance number), then the lot is accepted. Otherwise, it is rejected.

The action after rejection will be determined according to the contract made between the two parties involved. In this study, the rejection of a lot leads to paying an annoyance cost (a quality guarantee) to the consumer. That is, the producer will pay for not satisfying the requirements.

An inspection procedure has three basic cost components. These are the cost of inspection, the cost of outgoing defectives and the cost of rejection. The cost model selected in this study reflects these costs. In the cost function given below, if the number of defectives in a sample is less than or equal to the acceptance number, the lot is accepted, hence inspection cost is paid for the items in the sample and cost of defectives is paid for the outgoing defective items in the lot. If the number of defectives in the sample is greater than the acceptance number, than the lot is rejected; inspection cost is paid for the items in the sample, and an annoyance cost is paid for rejection.

$$TC = \begin{cases} n\,C_i + (N - n)\,p\,C_d & \text{if } d \le c \\ n\,C_i + C_A & \text{if } d > c \end{cases} \tag{3.3}$$

In a sampling plan, every lot has a predefined probability of acceptance and rejection. Here, the fundamental use of probability helps to find the chance of a lot passing sampling inspection. The operating characteristic (OC) curve represents the performance of a plan against different levels of quality (fraction defective) and as seen in Figure 1, the shape of OC curve changes as sample size (n) and acceptance number (c) change, and a given level of quality yields different risks for different sampling plans.

F(p) is the probability function of the chance of acceptance of a lot containing 100p % defectives for a given sample size n and acceptance number c (that is, given a sampling plan).

$$
\begin{aligned}
F(p) &= P(d \le c) \\
&= \sum_{r=0}^{c} \binom{n}{r} p^r (1-p)^{n-r} \\
&\cong \sum_{r=0}^{c} \frac{(n\,p)^r e^{-n\,p}}{r!} \qquad \text{for } n\,p > 5. \qquad (3.4).
\end{aligned}
$$

Throughout this study, Poisson approximation to binomial function is used as F(p).

The expected cost function of a sampling plan is given below.

$$
\begin{aligned}
E(TC) &= [n\,C_i + (N-n)\,p\,C_d]\,F(p) + [n\,C_i + C_A]\,[1 - F(p)] \\
&= C_A + n\,C_i + [(N-n)\,p\,C_d - C_A]\,F(p). \qquad (3.5)
\end{aligned}
$$

Then the suggested model combining both approaches can be expressed as

Min E(TC)

subject to

$\alpha \le \alpha_0$

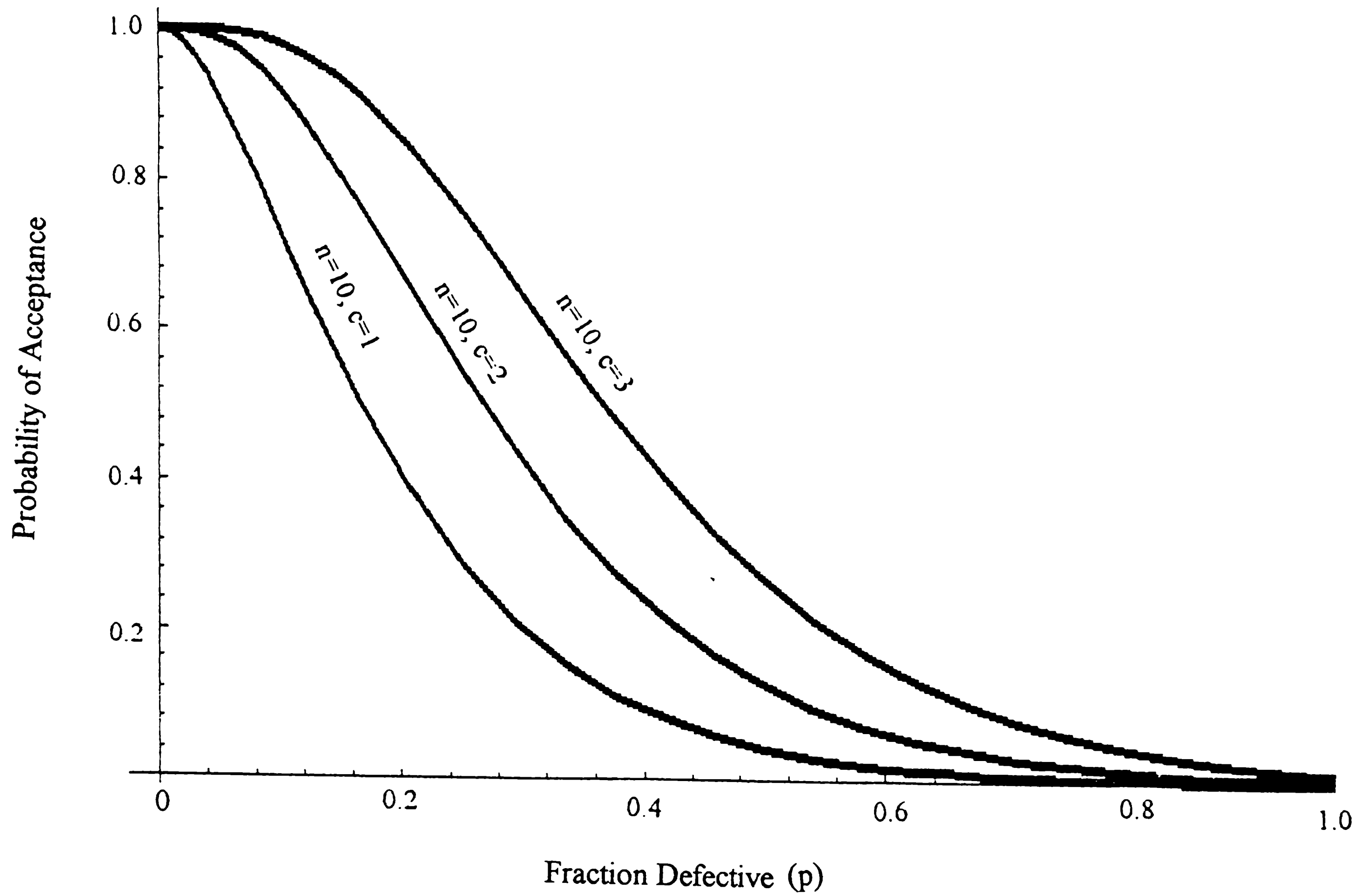

Figure 1. Operating Characteristic Curves for Different Sampling Plans

$\beta \leq \beta_0$

$n \leq N$

$c \leq n$

n, c : non-negative integer. (3.6)

and in the open form, the model is given below:

$$\text{Min } C_A + n\,C_i + [(N - n)\,p\,C_d - C_A] \sum_{r=0}^{c} \frac{(n\,p)^r e^{-n\,p}}{r!}$$

subject to

$$1 - \sum_{r=0}^{c} \frac{(n\,AQL)^r e^{-n\,AQL}}{r!} \leq \alpha_0$$

$$\sum_{r=0}^{c} \frac{(n\,LTPD)^r e^{-n\,LTPD}}{r!} \leq \beta_0$$

$n \leq N$

$c \leq n$

n, c : non-negative integer (3.7).

Although the model above is conceptually appealing, it is not very easy to solve. It has highly non-linear objective function and constraints. Besides, the decision variables are discrete. Hence, an efficient solution algorithm is required. As Nemhauser [15] has noted, regarding the methodology, integer nonlinear optimization is a ripe area and integer nonlinear software is practically non-existent. In the next chapter, a solution algorithm, specially designed to solve this problem, is presented.

3.2 Solution Method

A special two-stage algorithm is described to solve the suggested problem (to find the optimal sample size, n*, and the optimal acceptance number, c*) in this section.

At the first stage of this algorithm, the feasible region of the problem is determined. Given c, F(.) is a monotonic decreasing function of n (hence α increases and β decreases as n increases). With algebraic operations the constraints of the problem become

$$1 - \alpha \geq 1 - \alpha_0, \tag{3.8}$$

$$\beta \leq \beta_0. \tag{3.9}$$

For a given acceptance number c, the sample size should be less than or equal to a real number n^*_α, so that (3.1) is satisfied. Similarly, for a given c, n should be greater than or equal to a real number n^*_β to satisfy (3.2). Since consumer's and producer's risks affect the system in opposite directions, it is obvious that the feasible region may be empty for some set of parameters in this problem. There exists a feasible solution to the problem when

$$\lceil n^*_\beta \rceil \leq \lfloor n^*_\alpha \rfloor. \tag{3.10}$$

The stated relation is shown in the Figure 2.

For each c, if the bounds on n are known (that is, the feasible values of n are known), the problem can be rewritten as

$$\begin{aligned} &\text{Min } E(TC(n|c)) \\ &\text{subject to} \\ &\qquad \lceil n^*_\beta \rceil \leq n \leq \lfloor n^*_\alpha \rfloor \\ &\qquad n\text{: Integer} \end{aligned} \tag{3.11}$$

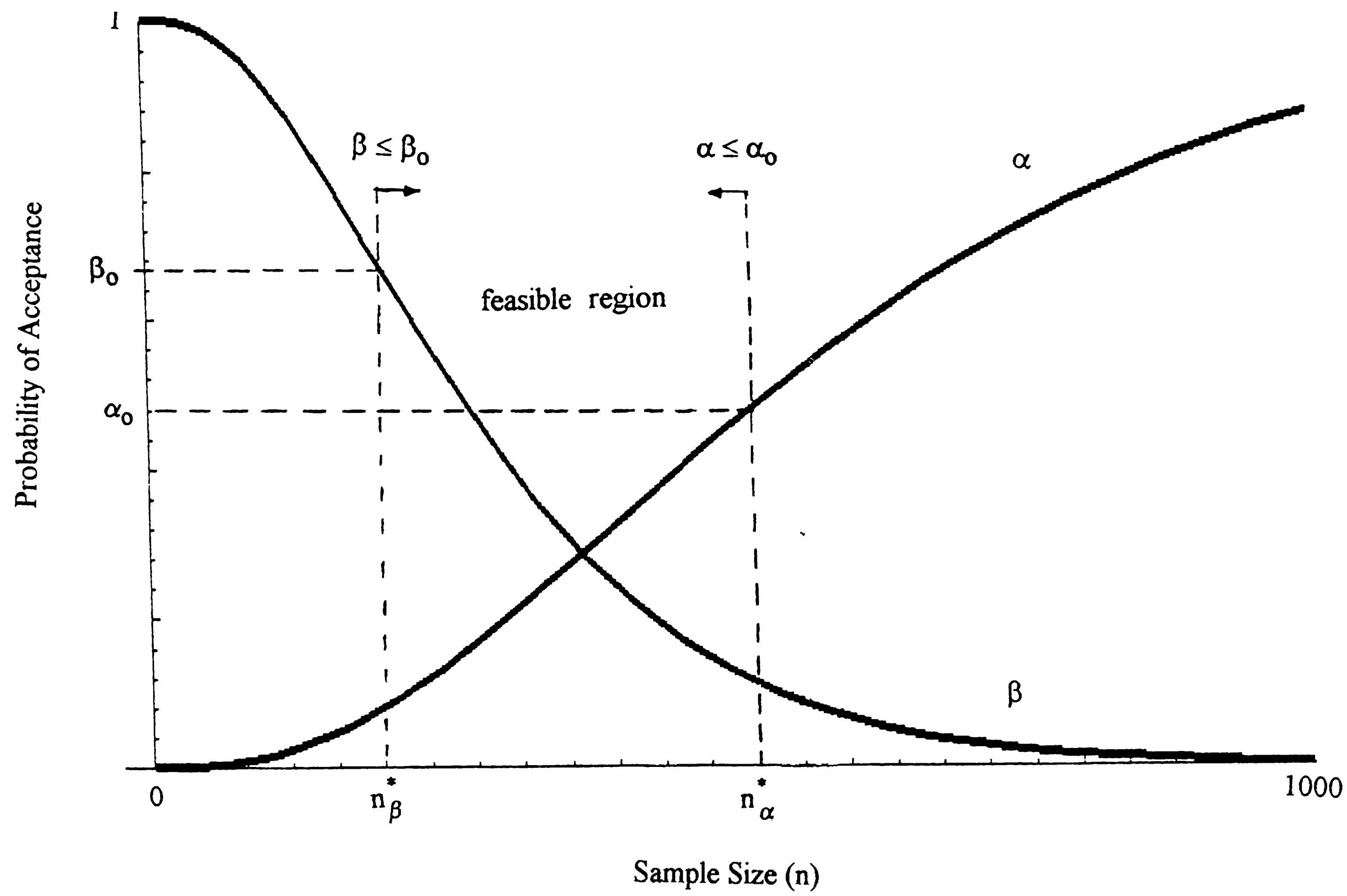

Figure 2. Feasible Region of Sample Size for Fixed Acceptance Number

where E(TC(n|c)) denotes that the acceptance number, c, is fixed in the formulation and the expected cost function is to be minimized over n only.

To compute $\lceil n^*_\beta \rceil$ and $\lfloor n^*_\alpha \rfloor$, it is possible to apply a simple binary search on the interval [0,N]. Binary search for $\lceil n^*_\beta \rceil$ can be summarized as follows:

Given the interval [0,N] for n, and given c,

Step 0: Let $n_1 = 0$, $n_2 = N$.

Evaluate β at $n = 0$ and $n = N$.

If $\beta(n = 0) < \beta_o$ then stop, $\lceil n^*_\beta \rceil = N$.

If $\beta(n = N) > \beta_o$ then stop, the problem is infeasible.

Step 1: If $\beta(n_1) \geq \beta_o \geq \beta(n_2)$, then let $n_3 = \lfloor \frac{n_1+n_2}{2} \rfloor$, and compute $\beta(n_3)$.

If $\beta(n_1) \geq \beta_o \geq \beta(n_3)$, then $n_2 \leftarrow n_3$.

If $\beta(n_3) \geq \beta_o \geq \beta(n_2)$, then $n_1 \leftarrow n_3$.

Step 2: Stop if $n_1 = n_2$, then $\lceil n^*_\beta \rceil = n_1 = n_2$.

Otherwise go to Step 1.

$\lfloor n^*_\alpha \rfloor$ can be determined in a similar fashion.

Even a 20-iteration binary search will be sufficiently precise on an interval $[0,2^{20}]$ which is large enough for the scope of this study.

As the objective function and the constraints of the problem are highly non-linear with respect to n, it is hard to find an optimal n even for a given c (even if c = 0, giving the simplest form of the objective function and the constraints, it is not easy to determine n* minimizing E(TC(n|c)). Non-linear programming techniques cannot be applied efficiently to get n* and c* simultaneously, since both variables are integers, the function is not differentiable and c is the variable determining the number of terms to be summed up in F(.). Besides, these techniques are not "reliable" in the sense that they depend too much on the initial conditions and may not yield the optimal solution all the time. Although the problem of discrete decision variables can be handled by rounding real valued solutions to integer, the problem of c determining the number of terms in F(.) can not be overcome easily. Unless c is fixed, numerical optimization procedures are not applicable. Of the mentioned difficulties, the former allows usage of available nonlinear optimization software, while the latter leads to inefficiencies in their usage.

Most appropriate way to find n* and c* seems to be finding n*(c) for each feasible c ($c \leq n$), and then enumerating each (n*(c), c) pair to determine n* and c* minimizing E(TC). For a very limited number of feasible n values, up to ten feasible points for instance, enumeration is an appropriate way of solving the problem, but for large feasible sets, it is extremely time consuming. Therefore, one needs to devise an algorithm that determines n*(c) quickly for a given c. This algorithm has to be fast since it is to be used several times in a single problem and n* is to be computed for all feasible c's.

Non-linearity of the constraints is not a problem, once the feasible set of n is determined as mentioned above. Non-linearity of the objective function, on the other hand, can be handled by using a piecewise linear approximation.

Let P1 and P2 be the values of F(p) evaluated at $\lceil n_\beta^* \rceil$ and $\lfloor n_\alpha^* \rfloor$, respectively as seen in Figure 3. Because F(p) decreases monotonically and displays an almost linear view between P1 and P2, F(p) may be approximated by a line passing through these two points. Assuming the function is continuous with respect to n (that is, n is

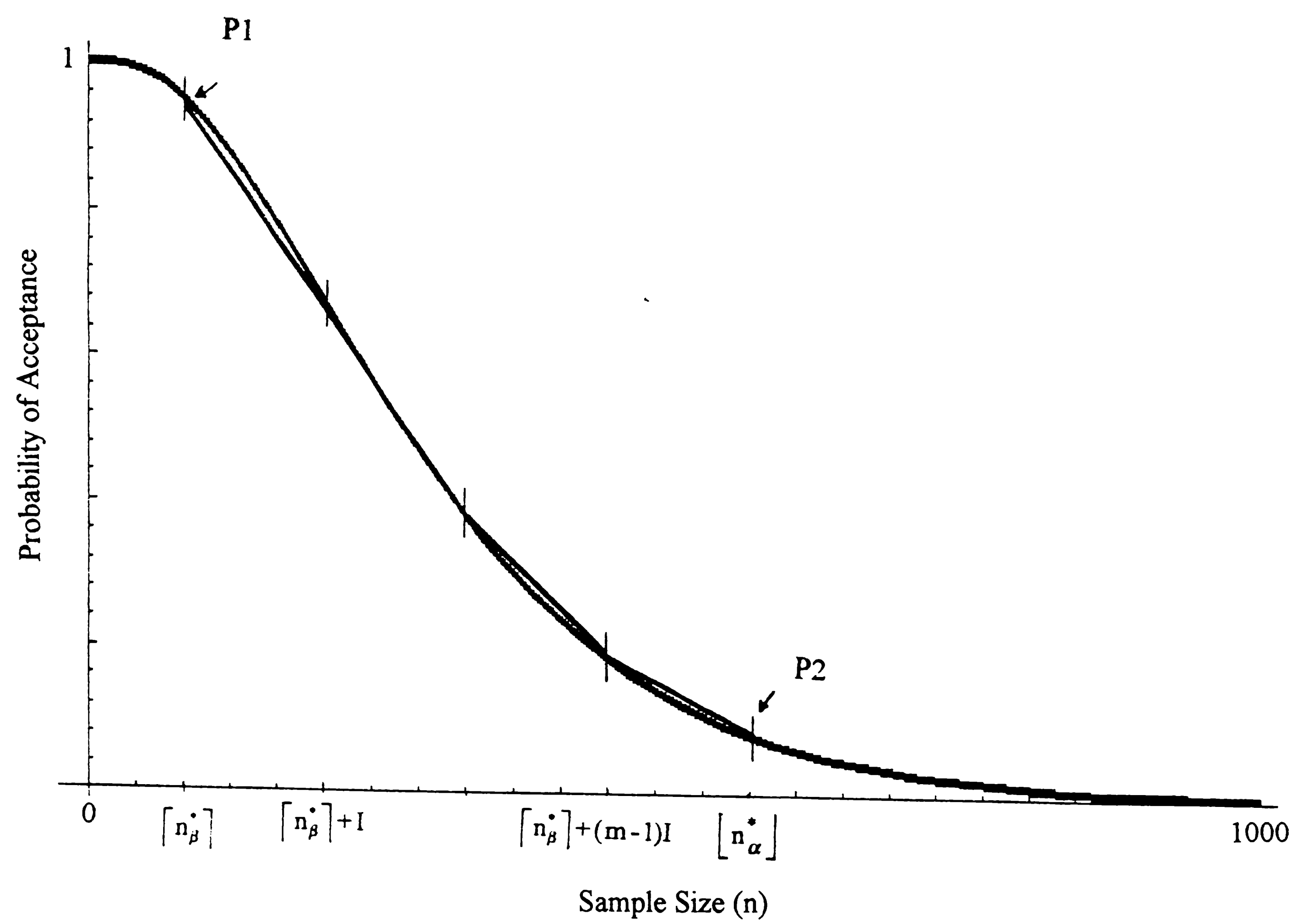

Figure 3. Piecewise Linear Approximation of OC Curve

real), a closer approximation can be obtained by piece-wise linearization of F(p). This linearization is performed by dividing the feasible region into a number of, m for instance, equal length intervals, such as,

$$[\lceil n_\beta^* \rceil, \lceil n_\beta^* \rceil + I], [\lceil n_\beta^* \rceil, \lceil n_\beta^* \rceil + 2I], \ldots, [\lceil n_\beta^* \rceil, \lceil n_\beta^* \rceil + mI]$$

where

$$I = \frac{\lfloor n_\alpha^* \rfloor - \lceil n_\beta^* \rceil}{m}. \tag{3.12}$$

By computing slopes (S_j, $j = 1,\ldots,m$) and intercepts (IN_j, $j = 1,\ldots,m$), the lines over the intervals are defined as follows :

$$S_j = \frac{F(p, \lceil n_\beta^* \rceil + j\,I) - F(p, \lceil n_\beta^* \rceil + (j-1)\,I)}{I}, \qquad j = 1,\ldots,m, \tag{3.13}$$

$$IN_j = F(p, \lceil n_\beta^* \rceil + j\,I) - S_i\,(\lceil n_\beta^* \rceil + j\,I), \qquad j = 1,\ldots,m. \tag{3.14}$$

The lower (L_j) and upper (U_j) end points of intervals can be expressed as

$$L_j = \lceil n_\beta^* \rceil + (j-1)\,I, \quad j = 1,\ldots,m, \tag{3.15}$$

$$U_j = \lceil n_\beta^* \rceil + j\,I \qquad j = 1,\ldots,m. \tag{3.16}$$

Let LF_j be the approximation of F(p) over the j^{th} interval and it is defined as

$$LF_j = S_j\,n + IN_j. \tag{3.17}$$

By replacing F(p) with its linear approximation, the new approximated E(TC(n|c)) over interval j becomes

$$E(TC(n|c))_j = C_A + C_i\, n + [(N - n)\, p\, C_d - C_A)\, LF_j \tag{3.18}$$

over the interval $O_j = [L_j, U_j]$.

Define n_j* as the optimal sample size over the j^{th} interval. Then, for a given c,

$$\text{Min } E(TC(n|c)) \equiv \underset{j=1,\ldots,m}{\text{Min}} \; \{ \text{Min } E(TC(n|c))_j \}. \tag{3.19}$$

A set of quadratic objective functions are obtained by the use of this linear approximation. Because the objective functions over the intervals become quadratic, it is simple to optimize these objective functions by use of calculus. The optimal sample size n_j* over the interval j will be determined by differentiating $E(TC(n|c))_j$ with respect to n for a given c value (assuming continuity of $E(TC(n|c))_j$ for n).

$$\frac{\partial}{\partial n} E(TC(n|c))_j = (\frac{C_i - IN_j\, p\, C_d}{S_j} + N\, p\, C_d - C_A)\, S_j - 2\, n\, p\, C_d\, S_j\, n = 0 \tag{3.20}$$

gives the minimum of the function over that interval since

$$\frac{\partial^2}{\partial n^2} E(TC(n|c))_j = -2\, p\, C_d\, S_j > 0 \text{ for all n.} \tag{3.21}$$

and F(p) being a decreasing function of n, $S_j < 0$. Therefore, $E(TC(n|c))_j$ is minimized at

$$n_j^* = \begin{cases} L_j & \text{if } \dfrac{C_i - IN_j pC_d + (NpC_d - C_A)S_j}{2pC_d S_j} < L_j \\ U_j & \text{if } \dfrac{C_i - IN_j pC_d + (NpC_d - C_A)S_j}{2pC_d S_j} > U_j \\ \dfrac{C_i - IN_j pC_d + (NpC_d - C_A)S_j}{2pC_d S_j} & \text{otherwise} \end{cases} \tag{3.22}$$

over O_j.

After determination of all n_j* over the intervals, the global optimal is found by comparing the minimum costs of the intervals as mentioned before.

Let $E(TC(n^*(c), c))$ define the minimum expected cost at $n^*(c)$ for a given acceptance number c. The optimal c* value is determined by finding the minimum $E(TC(n^*(c), c))$ for all feasible c.

Turbo Pascal has been used to code this solution method, and to test it as described in the next chapter.

4. TEST RESULTS

In this chapter, firstly the accuracy of the method proposed in the previous chapter is tested and then the results are presented and analyzed.

4.1. Problem Generation

A computer program is coded in Turbo Pascal to generate random problems. The problems are generated by determining values for nine parameters; C_i, C_d, C_A, N, AQL, LTPD, p, α_0, β_0. For convenience, the relations between parameters are defined as

$$C_i \leq C_d \leq C_A \leq 1000 \tag{4.1}$$

and

$$AQL \leq p \leq LTPD. \tag{4.2}$$

Three intervals are defined for the lot size, N; (0,1000], [1000,5000], and [5000,10000]. For each interval, a number of problems are generated and test are performed as will be discussed in the next section.

For the generation procedure, the built-in function, "RANDOM", of Turbo Pascal is used to find random values. This function returns value from uniform [0,1] distribution when called with no argument. If it is called with an argument, an integer value between 0 and that argument is returned.

In the code, first AQL is determined from uniform [0,1] distribution. The value obtained from uniform [0,1] distribution is multiplied by 0.1 and therefore, AQL is allowed to take values only in [0, 0.1]. Then, picking up two other random variables r_1 and r_2 from uniform [0,1] distribution, p and LTPD are determined:

$$LTPD = AQL\,(1 + 4\,r_1) \tag{4.3}$$

$$p = AQL + r_2\,(LTPD - AQL), \tag{4.4}$$

C_i, C_d, and C_A are determined in a similar fashion and they are not to violate (4.1).

α_0 and β_0 are determined in the same way AQL is obtained. For lot sizes between 0 and 1000, N is directly determined from the built-in function of Turbo Pascal. For N between 1000 and 5000, for instance, a random number between 0 and 4000 is obtained and 1000 is added to this value.

4.2. Accuracy and Efficiency of the Method

Two other computer programs coded in Turbo Pascal are used to test the accuracy of the algorithm. The first program is the implementation of the method and the second one is a program to find the optimal sampling plan by enumerating the feasible sample sizes and acceptance numbers. To reach a reliable implementation of the proposed method, the following has been observed, and related adjustments have been made.

(i) With the increase in the acceptance number, c, F(.) increases and approaches to one. Hence, for a large value of c, the change in F(.) will be negligible when the acceptance number is increased. But depending on the values of LTPD, AQL, p and n, the rate of convergence of F(.) to 1 changes. A graphical analysis is performed to determine a safe limiting number for c, since the higher the c, the more enumerations are required at the algorithm. Computation time increases drastically when c increases, and less and less the objective function is affected. For several problems with different sample sizes, F(.) is plotted as a function of fraction defective and acceptance number. It is observed that the value of F(.) is very close to 1 for acceptance numbers exceeding a number approximately equal to 15 % of the lot size, which gives the largest sample size for a problem. A number of figures are presented in Appendix A to illustrate this fact. Unfortunately, it has been experienced that even the solution time of problems with a lot size of 4000 is very high. Therefore, in the implementation of the algorithm, the limiting acceptance number is selected to be 50. If the optimal acceptance number determined by the method is very close or equal to this limiting number, then the problem is resolved for larger acceptance numbers. Although, this seems to decrease the efficiency of the algorithm, it is faster than the implementation of the 15% rule.

(i) The quality of the piecewise approximation used in the algorithm depends highly on the number of intervals, m, used. Efficiency of the algorithm also depends on m. Smaller the number of intervals, poorer is the approximation, but faster the solution is reached. When the approximation is poor, then the algorithm can not find exactly the minimum cost sampling plan, but returns sub-optimal plans. When the number of intervals in the approximation is high, then the piecewise linear function will look more like a continuous function, and the accuracy of the algorithm will increase, as well as the number of computations and the computation time.

To represent these observations, first the effect of m on the accuracy of the algorithm has been tested. For different lot sizes, the number of generated problems with optimal acceptance number less than 50 are tabulated in Table 1. The problems are chosen in this way, so that the proposed method will not require additional iterations and enumeration will not require too much computation. These problems have been solved by using different number of intervals for piecewise linear approximation in the proposed method, and by using enumeration. The sampling plans determined by enumeration and by the proposed method have been compared, and the number of problems for which the proposed method gives sub-optimal sampling plans are summarized in Table 2.

Table 1. Number of Generated Problems

Interval for N	No. of Problems
[0, 1000]	20
(1000, 2000]	5
(2000, 3000]	6
(3000, 4000]	5
(4000, 5000]	5
(5000, 6000]	4
(6000, 7000]	4
(7000, 8000]	4
(8000, 9000]	4
(9000, 10000]	5
TOTAL	62

Using the same set of problems, the computation time for different number of intervals in the approximation has been recorded, and the results are as seen in Figure 4.

Table 2. Number of Problems with Sub-optimal Sampling Plans

(out of 600 problems)

m	No. of Problems with Sub-optimal Solutions by Proposed Algorithm
1	8
5	6
10	4
15	2
20	2
30	0
40	0
50	0

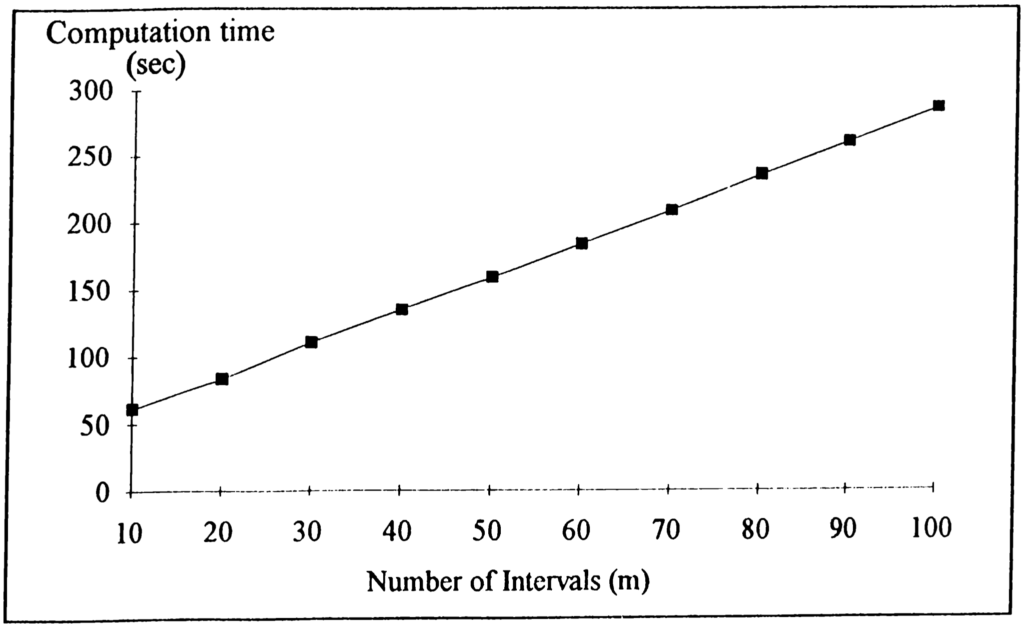

Figure 4. Change in Computation Time with Respect to m

Although, 50 seems to be an appropriate number of intervals in piecewise linear approximation, to be on the safe side, the number of intervals in the piecewise linearization procedure is taken as 100. When 100 intervals are used, the chance of getting a sub-optimal solution is fairly small, and the computation time is reasonable.

In testing the efficiency of the algorithm, computation times to solve ten randomly generated problems (with optimal c less than 50) by the proposed method, and the computation times to solve the same problems by enumeration have been recorded for ten different sets. The results are as given in Tables 3, 4, and 5.

Table 3. Computation Times (in seconds) to Solve 10 Randomly Generated Problems for N in [0, 1000] and m = 100

Problem Set	Number of Feasible Problems	Computation Time of Enumeration	Computation Time of Proposed Method
1	6	2.58	2.64
2	5	2.63	2.63
3	7	3.03	2.86
4	4	2.30	2.42
5	4	2.25	2.36
6	4	4.01	3.24
7	6	2.91	2.91
8	6	3.19	2.97
9	6	3.13	2.86
10	5	2.47	2.52
Average Computation Time		2.85	2.74

Table 4. Computation Times (in seconds) to Solve 10 Randomly Generated Problems for N in [1000, 5000] and m = 100

Problem Set	Number of Feasible Problems	Computation Time of Enumeration	Computation Time of Proposed Method
1	10	130.62	19.28
2	10	109.13	30.28
3	9	232.68	24.38
4	10	220.63	32.35
5	10	214.71	38.12
6	10	115.78	31.47
7	10	79.09	22.41
8	9	96.29	28.83
9	9	139.73	22.63
10	10	97.71	16.32
Average Computation Time		143.64	26.61

Table 5. Computation Times (in seconds) to Solve 10 Randomly Generated Problems for N in [5000, 10000] and m = 100

Problem Set	Number of Feasible Problems	Computation Time of Enumeration	Computation Time of Proposed Method
1	10	1334.09	75.57
2	10	1119.23	60.70
3	10	1685.06	61.29
4	9	832.73	50.92
5	10	1013.04	51.68
6	10	1343.59	51.03
7	10	2104.25	69.72
8	10	983.61	53.60
9	10	2664.32	91.07
10	10	1485.46	61.41
Average Computation Time		1456.54	62.70

As the lot size, N, increases, n can take values from a wider range, that is, the number of feasible points is likely to increase. When N is small, enumeration is nearly as fast as the proposed method. However, for large N, the computation time of the proposed method is almost "negligible" compared to that of enumeration. As a result of these tests, the claim that using piecewise linear approximation in solving the hybrid model increases efficiency is proved to be true.

CHAPTER 5. SENSITIVITY ANALYSIS OF THE MODEL

In this chapter, the sensitivity of the model with respect to problem parameters is analyzed. This analysis is performed in two ways. First, the effects of statistical constraints on the cost function is tested, and then the effects of parameters on the decision variables and on the objective function of the problem are discussed.

5.1. The Effect of Statistical Considerations on an Economical Model

The hybrid model presented in Chapter 3, combines two different approaches in the literature. As mentioned before, the economical models do not care much about the statistical consequences of their outcomes. Hybrid model puts two constraints on an economical model, and does not allow the consumer's and producer's risks to be high. In certain cases, a sampling plan determined solely by an economical model, may have low consumer's and producer's risks at given AQL and LTPD values and adding statistical constraints to the problem has no effect. However, in some others, the decision maker has to "pay" to have low risks for producer and consumer.

To detect the individual effects of risks, four problems have been solved first by ignoring the statistical constraints, and then by adding consumer's risk constraint and producer's risk constraint, separately. The optimal sampling plans for the unconstrained problem have been determined by the procedure discussed in [5]. The standard values 0.01, 0.05 and 0.10 are used as α_0 and β_0. In the first example, the optimal sampling plan, determined by the unconstrained problem, has a low risk for the consumer, but a high risk for the producer. Constraint on consumer's risk has no effect on the solution of the problem since specified β_0 values are already greater than the value of β of the optimal sampling plan of the unconstrained problem. As seen in Table 6, the lower the α_0 is, the more decision maker has to pay. To keep the producer's risk as low as 10%, the decision maker has to pay 17467.948 monetary units (MU) in addition to 3824.2 MU of the cost of economical model. Besides, as α_0 decreases (producer's risk is lower), then the optimal sampling plans are such that, β of the system increases (consumer's risk increases). The decision maker has to sacrifice from consumer's risk to achieve a reasonable producer's risk. Nevertheless, β is very low for the all cases analyzed in this problem.

Table 6. The Optimal Sampling Plans for the First Example

Problem Parameters	C_i = 34 MU, C_d = 710 MU, C_A = 943 MU, N = 993, p = 0.0626, AQL = 0.0411, LTPD = 0.0907
Solution of unconstrained problem	n* = 68, c* = 0, Min E[TC] = 3824.2,□ α = 0.939, β = 0.0021
Solution of the problem with α_0 = 0.01	n* = 572, c* = 35, Min E[TC] = 29109.337, □α = 0.0099, β = 0.0085
Solution of the problem with α_0 = 0.05	n* = 440, c* = 25, Min E[TC] = 24301.447, α = 0.0466, β = 0.0078
Solution of the problem with α_0 = 0.10	n* = 372, c* = 20, Min E[TC] = 21292.148, □α = 0.096, β = 0.0076

In the second problem, the solution of the unconstrained problem suggests a no inspection plan, with rejection of all lots. Then, there is no cost of outgoing defective items and no cost of inspection; the only cost incurred is the cost of quality guarantee. In this case, the producer's and consumer's risk are not well defined. When α_0 = 0.01, the only feasible sample size for c = 0 in the problem is n* = 0 and the risks are not defined again. In other cases, the inverse relation between consumer's risk and producer's risk can be seen. To achieve a certain level in one of the risk factors, the decision maker has to sacrifice from the other. For this problem, achieving a certain producer's risk is more costly than achieving a level of consumer's risk. The results are given in Table 7.

Table 7. The Optimal Sampling Plans for the Second Example

Problem Parameters	C_i = 409 MU, C_d = 581 MU, C_A = 695 MU, N = 279, p = 0.1230, AQL = 0.0659, LTPD = 0.2662
Solution of unconstrained problem	n* = 0, c* = 0, Min E[TC] = 695 (the lots are always rejected)
Solution of the problem with α_0 = 0.01	n* = 0, c* = 0, Min E[TC] = 695 (the lots are always rejected)
Solution of the problem with α_0 = 0.05	n* = 5, c* = 1, Min E[TC] = 19229.908, α = 0.044, β = 0.616
Solution of the problem with α_0 = 0.10	n* = 8, c* = 1, Min E[TC] = 17814.599, α = 0.098, β = 0.372
Solution of the problem with β_0 = 0.01	n* = 18, c* = 0, Min E[TC] = 10019.014, α = 0.694, β = 0.0082
Solution of the problem with β_0 = 0.05	n* = 14, c* = 0, Min E[TC] = 9681.120, α = 0.397, β = 0.024
Solution of the problem with β_0 = 0.10	n* = 14, c* = 0, Min E[TC] = 9681.120, α = 0.397, β = 0.024

To test the interactive effects the constraints, the experiment in Table 8 is performed. For fixed α_0, the cost increases as β_0 decreases, and for fixed β_0, the cost and α_0 have an inverse relationship. The interactive effects of β_0 and α_0 do not seem to be systematic in this example.

In the third problem, since the cost of defective items is very small, the lots are accepted without any inspection. Then the cost is determined by the expected cost of

Table 8. Interactive Effects of Constraints for the Second Problem

Problem Parameters	C_i = 409 MU, C_d = 581 MU, C_A = 695 MU, N = 279, p = 0.1230, AQL = 0.0659, LTPD = 0.2662
Solution of unconstrained problem	n* = 0, c* = 0, Min E[TC] = 695 (the lots are always rejected)
Solution of the problem with $\alpha_0 = 0.01$, $\beta_0 = 0.01$	n* = 81, c* = 11, Min E[TC] = 43255.410 $\alpha = 0.0088$, $\beta = 0.0096$
Solution of the problem with $\alpha_0 = 0.01$, $\beta_0 = 0.05$	n* = 60, c* = 9, Min E[TC] = 37049.341 $\alpha = 0.0074$, $\beta = 0.04389$
Solution of the problem with $\alpha_0 = 0.01$, $\beta_0 = 0.10$	n* = 49, c* = 8, Min E[TC] = 34028.697 $\alpha = 0.0061$, $\beta = 0.0977$
Solution of the problem with $\alpha_0 = 0.05$, $\beta_0 = 0.01$	n* = 66, c* = 8, Min E[TC] = 36057.507 $\alpha = 0.0337$, $\beta = 0.0091$
Solution of the problem with $\alpha_0 = 0.05$, $\beta_0 = 0.05$	n* = 45, c* = 8, Min E[TC] = 30007.060 $\alpha = 0.032$, $\beta = 0.0463$
Solution of the problem with $\alpha_0 = 0.05$, $\beta_0 = 0.10$	n* = 35, c* = 5, Min E[TC] = 27329.263 $\alpha = 0.031$, $\beta = 0.0978$
Solution of the problem with $\alpha_0 = 0.10$, $\beta_0 = 0.01$	n* = 55, c* = 6, Min E[TC] = 30620.953 $\alpha = 0.075$, $\beta = 0.0096$
Solution of the problem with $\alpha_0 = 0.10$, $\beta_0 = 0.05$	n* = 36, c* = 4, Min E[TC] = 24518.133 $\alpha = 0.093$, $\beta = 0.038$
Solution of the problem with $\alpha_0 = 0.10$, $\beta_0 = 0.10$	n* = 26, c* = 3, Min E[TC] = 21811.630 $\alpha = 0.095$, $\beta = 0.086$

outgoing defective items (since this cost is less than the cost of rejection, the lots are accepted). When a bound on the producer's risk is imposed, the optimal sampling plan is again defined by this policy. As expected, as the bound on consumer's risk is tightened (β_0 decreases), the cost increases. The decision maker has to pay at least an additional 188.666 MU, to achieve a risk less than 0.10 for the consumer. The results are tabulated in Table 9.

Table 9. The Optimal Sampling Plans for the Third Example

Problem Parameters	C_i = 2 MU, C_d = 2 MU, C_A = 511 MU, N = 2407, p = 0.0736, AQL = 0.0444, LTPD = 0.087
Solution of unconstrained problem	n* = 0, c* = 0, Min E[TC] = 354.310 (the lots are always accepted)
Solution of the problem with α_0 = 0.01	n* = 0, c* = 0, Min E[TC] = 354.310 (the lots are always accepted)
Solution of the problem with α_0 = 0.05	n* = 0, c* = 0, Min E[TC] = 354.310 (the lots are always accepted)
Solution of the problem with α_0 = 0.10	n* = 0, c* = 0, Min E[TC] = 354.310 (the lots are always accepted)
Solution of the problem with β_0 = 0.01	n* = 53, c* = 0, Min E[TC] = 613.673, □α = 0.905, β = 0.0099
Solution of the problem with β_0 = 0.05	n* = 35, c* = 0, Min E[TC] = 568.687, □α = 0.789, β = 0.0476
Solution of the problem with β_0 = 0.10	n* = 27, c* = 0, Min E[TC] = 542.976 □α = 0.698, β = 0.0954

As displayed in Table 10, the interactive effects of producer's and consumer's risks on the economical model are not systematic for the third example, too. In this example, it is beneficial to keep n and c as small as possible, but the constraints force the decision variables to take non-zero value. One observation in the examples solved to test the interactive effects is that, n takes its minimum allowable value, for all given c*'s.

Table 10. Interactive Effects of Constraints for the Third Problem

Problem Parameters	C_i = 2 MU, C_d = 2 MU, C_A = 511 MU, N = 2407, p = 0.0736, AQL = 0.0444, LTPD = 0.087
Solution of unconstrained problem	n* = 0, c* = 0, Min E[TC] = 354.310 (the lots are always accepted)
Solution of the problem with $\alpha_0 = 0.01$, $\beta_0 = 0.01$	n* = 768, c* = 48, Min E[TC] = 2008.702, $\alpha = 0.0095$, $\beta = 0.0098$
Solution of the problem with $\alpha_0 = 0.01$, $\beta_0 = 0.05$	n* = 547, c* = 36, Min E[TC] = 1537.977, $\alpha = 0.0097$, $\beta = 0.0493$
Solution of the problem with $\alpha_0 = 0.01$, $\beta_0 = 0.10$	n* = 454, c* = 31, Min E[TC] = 1334.039, $\alpha = 0.0090$, $\beta = 0.0981$
Solution of the problem with $\alpha_0 = 0.05$, $\beta_0 = 0.01$	n* = 578, c* = 34, Min E[TC] = 1641.393, $\alpha = 0.046$, $\beta = 0.0097$
Solution of the problem with $\alpha_0 = 0.05$, $\beta_0 = 0.05$	n* = 388, c* = 24, Min E[TC] = 1238.298, $\alpha = 0.046$, $\beta = 0.0499$
Solution of the problem with $\alpha_0 = 0.05$, $\beta_0 = 0.10$	n* = 298, c* = 19, Min E[TC] = 1044.629, $\alpha = 0.0493$, $\beta = 0.0992$
Solution of the problem with $\alpha_0 = 0.10$, $\beta_0 = 0.01$	n* = 480, c* = 27, Min E[TC] = 1450.565, $\alpha = 0.094$, $\beta = 0.0099$
Solution of the problem with $\alpha_0 = 0.10$, $\beta_0 = 0.05$	n* = 307, c* = 18, Min E[TC] = 1085.28, $\alpha = 0.098$, $\beta = 0.049$
Solution of the problem with $\alpha_0 = 0.10$, $\beta_0 = 0.10$	n* = 245, c* = 15, Min E[TC] = 946.23, $\alpha = 0.086$, $\beta = 0.099$

Table 11 summarizes the results of sensitivity analysis for the fourth example. A bound imposed by α_0 changes the optimal sampling plan, but does not affect the cost. This indicates that F(p) is very close to 1, practically when c = 146, and there is almost no change in F(p) even though c changes and takes values between 146 and 850. The relation between β_0 and cost is as mentioned in the previous example.

Table 11. The Optimal Sampling Plans for the Fourth Example

Problem Parameters	C_i = 1 MU, C_d = 20 MU, C_A = 1500 MU, N = 850, p = 0.0736, AQL = 0.0444, LTPD = 0.087
Solution of unconstrained problem	n* = 850, c* = 850, Min E[TC] = 850.0 (the lots are always accepted)
Solution of the problem with α_0 = 0.01, α_0 = 0.05, α_0 = 0.10	n* = 850, c* = 146, Min E[TC] = 850.0 (the lots are always accepted)
Solution of the problem with β_0 = 0.01	n* = 53, c* = 0, Min E[TC] = 1546.390, □α = 0.905, β = 0.0099
Solution of the problem with β_0 = 0.05	n* = 35, c* = 0, Min E[TC] = 1512.152, □α = 0.789, β = 0.0476
Solution of the problem with β_0 = 0.10	n* = 27, c* = 0, Min E[TC] = 1487.447, □α = 0.698, β = 0.0954

Results of the test regarding the interactive effect of constraints is presented in Table 12. The producer's risk decreases as the bound on consumer's risk is loosened, and this was a result observed in the previous examples, too.

The effect of constraints on the problem should in fact be assessed with the effects of other statistical parameters, namely, LTPD and AQL, as well as others

Table 12. Interactive Effects of Constraints for the Fourth Problem

Problem Parameters	C_i = 1 MU, C_d = 20 MU, C_A = 1500 MU, N = 850, p = 0.0736, AQL = 0.0444, LTPD = 0.087
Solution of unconstrained problem	n* = 850, c* =850, Min E[TC] =850 (the lots are always accepted)
Solution of the problem with $\alpha_0 = 0.01$, $\beta_0 = 0.01$	n* = 768, c* = 48, Min E[TC] = 2072.167, α = 0.0095, β =0.0098
Solution of the problem with $\alpha_0 = 0.01$, $\beta_0 = 0.05$	n* = 547, c* = 36, Min E[TC] = 1749.197, α = 0.0097, β = 0.0493
Solution of the problem with $\alpha_0 = 0.01$, $\beta_0 = 0.10$	n* = 844, c* = 62, Min E[TC] = 1557.085, α =0.00009 , β =0.0987
Solution of the problem with $\alpha_0 = 0.05$, $\beta_0 = 0.01$	n* = 578, c* = 34, Min E[TC] = 1961.533, α =0.046 , β = 0.0097
Solution of the problem with $\alpha_0 = 0.05$, $\beta_0 = 0.05$	n* = 388, c* = 24, Min E[TC] = 1701.228, α = 0.046, β =0.0499
Solution of the problem with $\alpha_0 = 0.05$, $\beta_0 = 0.10$	n* = 844, c* = 62, Min E[TC] = 1557.085, α =0.00009 , β =0.0987
Solution of the problem with $\alpha_0 = 0.10$, $\beta_0 = 0.01$	n* = 480, c* = 27, Min E[TC] = 1894.128, α = 0.094, β = 0.0099
Solution of the problem with $\alpha_0 = 0.10$, $\beta_0 = 0.05$	n* = 307, c* = 18, Min E[TC] = 1669.136, α = 0.098, β =0.049
Solution of the problem with $\alpha_0 = 0.10$, $\beta_0 = 0.10$	n* = 844, c* = 62, Min E[TC] = 1557.085, α =0.00009 , β =0.0987

parameters. However, the number of parameters being high, an experiment to test their effects on the decision variables is almost impossible to devise. The effect of LTPD and AQL, on the other hand, can be inferred from their statistical implications.

The higher the LTPD, the lower the consumer's risk for fixed n and c, and as seen in Figure 1, when c and LTPD are fixed, the consumer's risk increases as n increases. The effects of LTPD and AQL on the risk of consumer and producer, respectively, may be combined with the observations above, and the sensitivity of the model may be deduced with respect to these parameters. In the following section, the individual effects of the parameters have been investigated.

5.2. The Effect of Parameters

In this section a base problem is solved several times, and the effects of cost of inspection, cost of defective items, cost of annoyance, lot size and process fraction defective are tested. The tests here are focused more on the economical considerations, rather than the statistical ones, hence LTPD, AQL, α_0 and β_0 are kept as constants. These parameters only affect the feasible region of the problem, and the main concern here is to understand how optimal sampling plans in a given feasible region change with changing economical components of the model.

Additionally, two cost indicators (C_i-pC_d) and (NpC_d-C_A) are considered in this section. The change in these cost indicators are observed in the analysis. Here, (C_i-pC_d) gives us the difference between the cost of inspection per unit and the expected cost of an outgoing item. In fact, this difference is the opportunity cost between inspecting an item and passing the item with no inspection. It is expected to have less inspection when this indicator is positive and vice versa. The other cost indicator (NpC_d-C_A) is the opportunity cost of sending a lot to customer with no inspection or rejecting the lot. In the contrary, the positive values of this indicator leads to expectedly larger sample size and vice versa.

5.2.1. The Effect of Cost of Inspection

To analyze the effect of cost of inspection, the original problem is solved for different values of C_i. The changes in E[TC] and the cost indicators (C_i-pC_d) and (NpC_d-C_A) are observed. As expected, with the increase in the cost of inspection, the total expected cost is increased. For negative values of the cost indicator (C_i-pC_d), the model selects plans that have sample sizes as large as possible (Note that the negative values of (C_i-pC_d) indicate higher expected cost of outgoing defective units). That is, the model prefers to inspect as much as it can, in order not to pay the cost of defectives.

For high cost of inspection, the model selects sampling plans with the lowest possible sample size, as expected. As shown in the table, the change in the cost of inspection per unit affects significantly the selection of the model. The model is highly sensitive to the change in this parameter. If the value of C_i is increased from 1 to 5, the new cost is 1.5 times the previous one. Therefore, in the estimation of this parameter, very accurate models should be used.

Table 13. Change in Cost of Inspection

C_i	n*	c*	Min E[TC]	$C_i - p\,C_d$	$N\,p\,C_d - C_A$
0.01	589	50	412.733	-39.134	33257.84
0.1	511	44	461.366	-39.044	33257.84
1	383	34	841.625	-38.144	33257.84
5	294	27	2155.54	-34.144	33257.84
10	257	24	3510.765	-29.144	33257.84
20	208	20	5814.345	-19.144	33257.84
50	148	15	11207.59	10.856	33257.84
100	113	12	17670.72	60.856	33257.84
200	58	7	25576.5	160.856	33257.84
500	58	7	42976.5	460.856	33257.84

5.2.2. The Effect of Cost of Defectives

In a similar way, the effect of cost of defectives are analyzed. The increase in the cost of defectives increases the total expected cost while decreasing the sample size. The acceptance number increases as the cost of defectives increases. The change in the total expected cost due to the change in the cost of defectives is less than that of the cost of inspection.

As it is seen in (3.5), the cost of inspection has a direct impact on the change in the expected total cost function, when compared with the cost of defectives. The effect of C_d on E(TC) depends highly on the value of the average process fraction defective (p) and the probability of accepting the lot (F(p)), because its product with p and F(p) is incurred in E(TC). But, as shown in the Table 14, the change in this parameter affects the selection of the model significantly, as well. Thus, the model is also very sensitive to this parameter.

Table 14. Change in Cost of Defective Items

C_d	n*	c*	Min E[TC]	C_i - p C_d	N p C_d - C_A
20	58	7	2370.22	17.204	1998.56
50	90	10	3645.626	13.01	5605.4
200	195	19	5445.044	-7.96	23639.6
300	220	21	5903.374	-21.94	35662.4
400	232	22	6250.291	-35.92	47685.2
500	257	24	6508.291	-49.9	59708

5.2.3. The Effect of Cost of Annoyance

Cost of annoyance is the fixed cost that the producer will pay in case of rejection. With the increase in the probability of rejection, the effect of this parameter will be obvious. The cost of annoyance affects the value of (NpC_d-C_A).

It is observed in the Table 15 that the model switch from one optimal to another while the value of C_A increases. For small values of it (large values of (NpC_d-C_A)), the model selects the sampling plan n*=208 and c*=20. Although the value of the cost of defectives increased about 20 fold, the optimal sampling plan does not change. For the values of C_A greater than 12370, the model switch to the sampling plan n*=860 and c*=180, where n* gets equal to lot size and c* is very high so that F(p) is very close to 1. Thus, the model is very robust on the cost of annoyance. The errors made in estimating of the value of C_A will not affect the optimal sampling plan of our model.

Table 15. Change in Cost of Annoyance

C_A	n*	c*	Min E[TC]	C_i - p Cd	N p Cd - C_A
50	208	20	5476.04	-19.144	33613.84
100	208	20	5523.555	-19.144	33563.84
200	208	20	5618.584	-19.144	33463.84
300	208	20	5713.614	-19.144	33363.84
400	208	20	5808.644	-19.144	33263.84
500	208	20	5903.673	-19.144	33163.84
1000	208	20	6378.822	-19.144	32663.84
12370	208	20	17199.967	-19.144	2664.84
12371	860	180	17200.01	-19.144	2663.84
31000	860	180	17200.01	-19.144	2663.84
31500	860	180	17200.01	-19.144	2163.84
32000	860	180	17200.01	-19.144	1663.84
32500	860	180	17200.01	-19.144	1163.84
33000	860	180	17200.01	-19.144	663.84
34000	860	180	17200.01	-19.144	-336.16
34500	860	180	17200.01	-19.144	-836.16
35000	860	180	17200.01	-19.144	-1336.16
36000	860	180	17200.01	-19.144	-2336.16
37000	860	180	17200.01	-19.144	-3336.16

5.2.4. The Effect of Lot Size

The change in the lot size directly affects the feasible region of sample size and thus the optimal values. But it also affects the cost indicator (NpC_d-C_A). When this cost indicator is positive, the expected cost increases with the increase in the lot size because, as mentioned in the previous section, the model tries to inspect as much as possible. The sample size increases as lot size increases. The optimal sampling plan and its cost is tabulated for different values of the lot size in the Table 16.

It is obvious that the model is very sensitive the change in the lot size. Because the change in lot size affects the feasible region, the probability acceptance of a lot is highly dependent on this parameter. If the new value of lot size makes the optimal sample size of previous problem, the problem is forced to another optimal point.

Table 16. Change in Lot Size

N	n*	c*	Min E[TC]	$C_i - p\,C_d$	$N\,p\,C_d - C_A$
100	68	8	2097.054	-19.144	3508.4
200	113	12	3286.482	-19.144	7422.8
300	136	14	4022.59	-19.144	11337.2
400	148	15	4537.447	-19.144	15251.6
500	171	17	4949.437	-19.144	19166
600	183	18	5261.006	-19.144	23080.4
800	208	20	5697.61	-19.144	30909.2
1000	220	21	6056.58	-19.144	38738

5.2.5. The Effect of Process Fraction Defective

The probability of acceptance is also affected by the change in the process fraction defective. This parameter has direct relation with the probability of acceptance, when compare to that of lot size. The effect of this parameter on the expected total cost depends high on the values of other parameter, because it has direct effect on the cost indicators, as shown in Table 17, as well. Furthermore, as mention in the previous chapter before, its value depends on the statistical parameters AQL and LTPD.

The trade-off between inspecting an item or letting it to go out is seen in Table 17. The increase in the process fraction defective makes the sign of the cost indicator

(C_i-pC_d) negative. Thus, inspecting an item becomes cheaper than the expected cost of an outgoing defective item (pC_d). With this sign change, the optimal sample starts to increase with the increase in the process fraction defective.

As it will seen in the table, for the values of p less then 0.054, the model selects the sampling plan n*=57, c*=7. Therefore, it can be concluded that it can be found a robust interval for this parameter. But, the model is significantly sensitive for other values.

Table 17. Change in Process Fraction Defective

p	n*	c*	Min E[TC]	$C_i - p\,C_d$	$N\,p\,C_d - C_A$
0.0014	57	7	1454.776	19.608	-68.88
0.0015	57	7	1477.26	19.58	-44.8
0.0017	57	7	1522.228	19.524	3.36
0.0018	57	7	1544.712	19.496	27.44
0.0024	57	7	1679.616	19.328	171.92
0.0034	57	7	1904.456	19.048	412.72
0.0044	57	7	2129.296	18.768	653.52
0.0054	57	7	2354.136	18.488	894.32
0.0064	57	7	2578.976	18.208	1135.12
0.0071	57	7	2736.364	18.012	1303.68
0.0081	57	7	2961.204	17.732	1544.48
0.0091	57	7	3186.044	17.452	1785.28
0.014	57	7	4287.754	16.08	2965.2
0.024	57	7	6535.706	13.28	5373.2
0.034	57	7	8778.049	10.48	7781.2
0.044	57	7	10991.91	7.68	10189.2
0.054	58	7	13110.26	4.88	12597.2
0.064	58	7	15031.83	2.08	15005.2
0.0714	148	15	16208.08	0.008	16787.12
0.0814	435	38	15652.08	-2.792	19195.12
0.0914	396	35	13658.65	-5.592	21603.12

6. CONCLUSION

In this thesis, an acceptance sampling model combining two different approaches in the literature has been analyzed. As it would be noticed in Chapter 2, very few attempts have been made to combine the statistical and economical considerations of acceptance sampling plans, in optimization context.

The model presented in Chapter 3, is very appealing conceptually but not easy to solve. One of the main considerations in this study has been to develop an efficient solution algorithm to solve this nonlinear, integer mathematical programming problem. The proposed algorithm makes use of piecewise linearization of OC curves, and thus the objective function of the problem reduces to a quadratic form, and the problem is simplified, yet it is still complicated. Since this piecewise linearization is an approximation to the original problem, the "quality" of this approximation is tested by comparing the solution of a number of problems by this method to the solution of the problems by complete enumeration. As expected, it has been observed that the solutions of the proposed method are closer to that of enumeration as the number of intervals in the linearization procedure increases.

In addition to the tests on the proposed algorithm, sensitivity analysis on the hybrid model is performed, and the effects of statistical constraints on the economical objective function are tested. On different examples it has been shown that, depending on the parameters of the problem, the "cost" of statistical considerations may be very high or none at all. Sensitivity of the problem is also tested with respect to individual parameters. The followings are observed in this analysis:

i) the model is robust for some of the parameters while it is very sensitive to the change in others

ii) interactive effects of parameters should be analyzed because there exists the products of parameters in the model. But complete analysis of the model is very difficult due to the vast number of parameters.

iii) depending on the parameters, the statistical constraints may have no effect on the economical model. For other set of parameters, constraints affect the selection of the model.

Only one of the cost models existing in the literature is used in this study. But other cost models can be easily adapted to the model as discussed in [5].

Acceptance sampling is an indirect way of the quality control. In fact, it does not improve the quality of process used in the production. Furthermore, the quality level is a term used to define an acceptable percentage of defectives in the lot. The quality control approaches are changed from the traditional quality level approaches to measurement in term of the defects in part per million, even in billion [11]. In fact, to achieve high quality, these percentages (AQL and LTPD in our model) should be very close to zero. Moreover, there are some production areas that the quality level concept is not compatible. In such areas, the cost of defective is extremely high or it is impossible to estimate. For example, one can not apply the quality level approaches in the health services.

In order to improve the quality and to achieve zero defectives the source of defectives should be detected [17]. But sampling only detects the defective items and does not give any information on their source to correct. Thus, it provides only the quality protection.

The effective quality control should be able to combine statistical and economical aspects with the idea of zero defectives. Statistics and economics should be the means to reduce the number of defectives and improve the quality. A new approach has to be developed to account for these feature of quality control. In any case, the proposed model can be used to see the effects of zero defectives on the cost function by letting the p, AQL, and LTPD to take very small values.

Quality control is an important subject of a world suffering from the scarcity. Wasting any kind of resource (material, labor, or time) for low quality products or services is very crucial for all societies and high quality should be aimed in all cases in order to lower the cost to the societies. Thus, the tools used for quality control should be effective.

The model contains the cost of both producer and consumer in E(TC). In some cases, the model selection may increase the cost to producer and in some cases, cost to consumer. Therefore, another problem arises from combining the statistical and economical approaches. This problem can be encompassed by means of some rules regulating these cases.

In the hybrid model, fraction defectives were assumed to be deterministic. In some cases, it is not possible to know the exact value of the fraction defective, but, the distribution of fraction defective can be known. As a future research direction, a model may be developed to combine Bayesian, economical, and statistical approaches with the use of the distribution of the fraction defective. Without any doubt, such a model would be more realistic, but harder than the hybrid model to solve, which makes it more challenging.

Another way to combine the statistical and economical aspects of an acceptance sampling plan is to have the statistical expressions in the objective function and the economical one in the constraint, such as a model minimizing the producer's and consumer's risks subject to a budget constraint. This kind of a hybrid model would again require an efficient solution method.

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APPENDIX

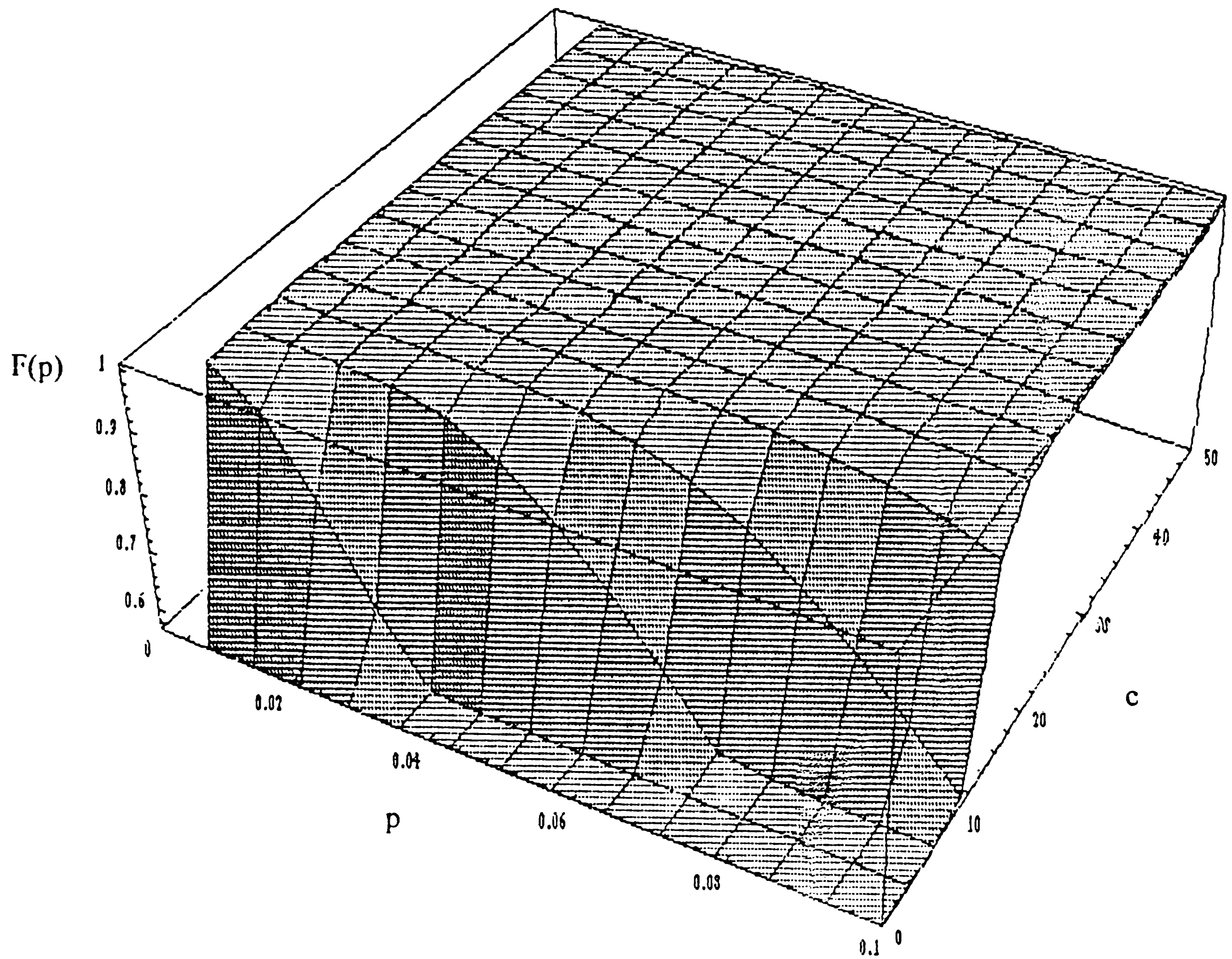

Figure A.1. F(p) as a Function of p and c, when n = 100

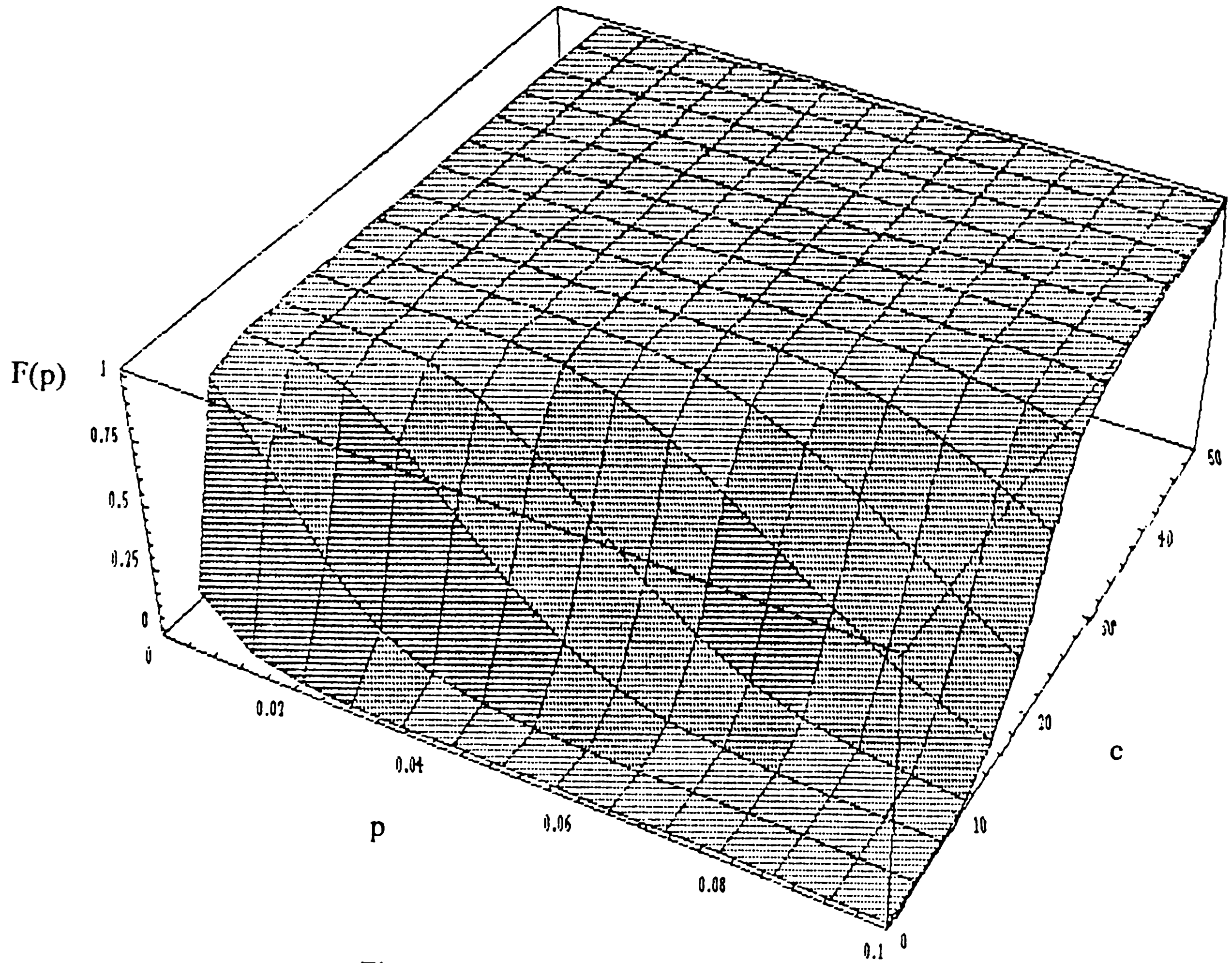

Figure A.2. F(p) as a Function of p and c, when n = 200

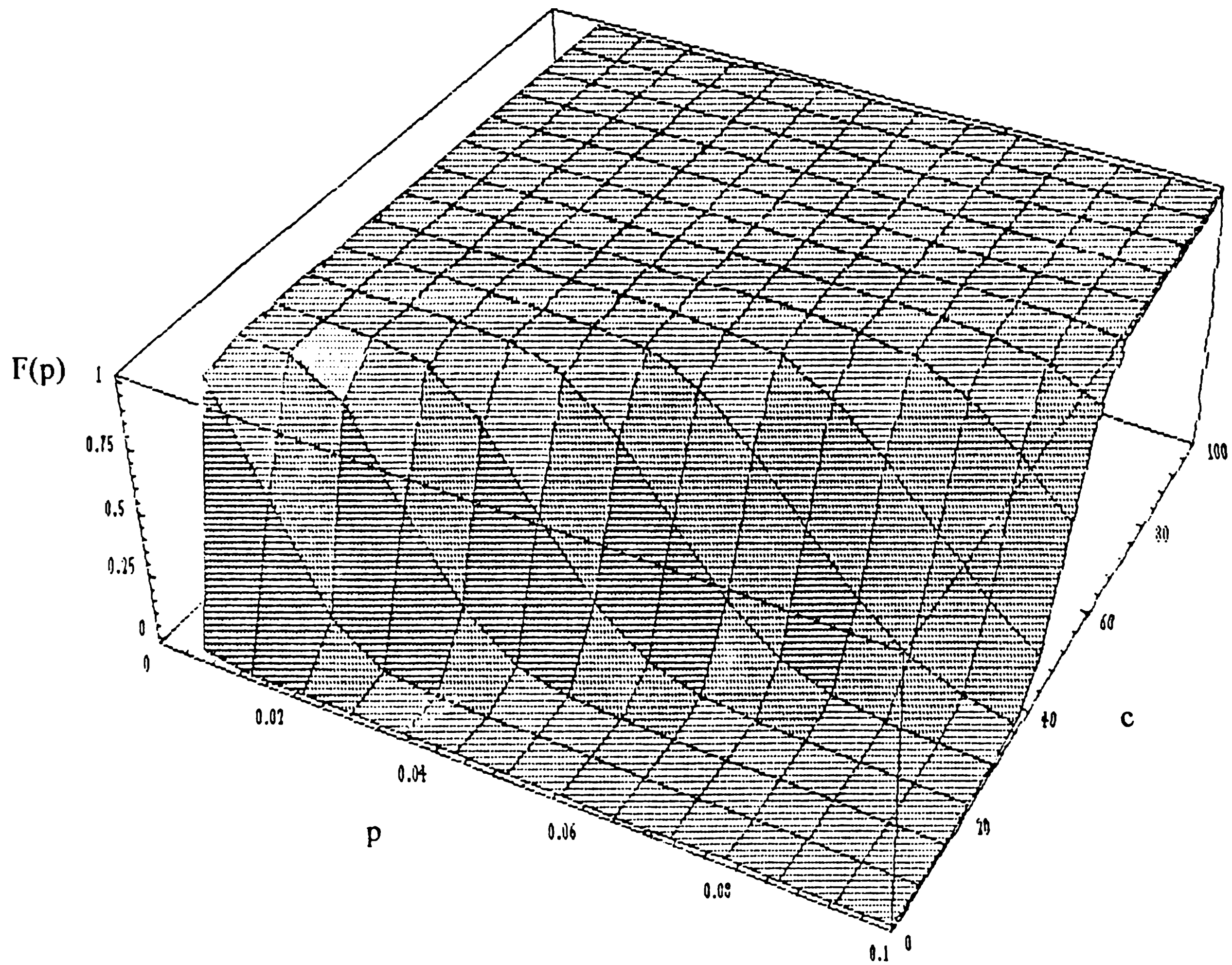

Figure A.3. F(p) as a Function of p and c, when n = 500

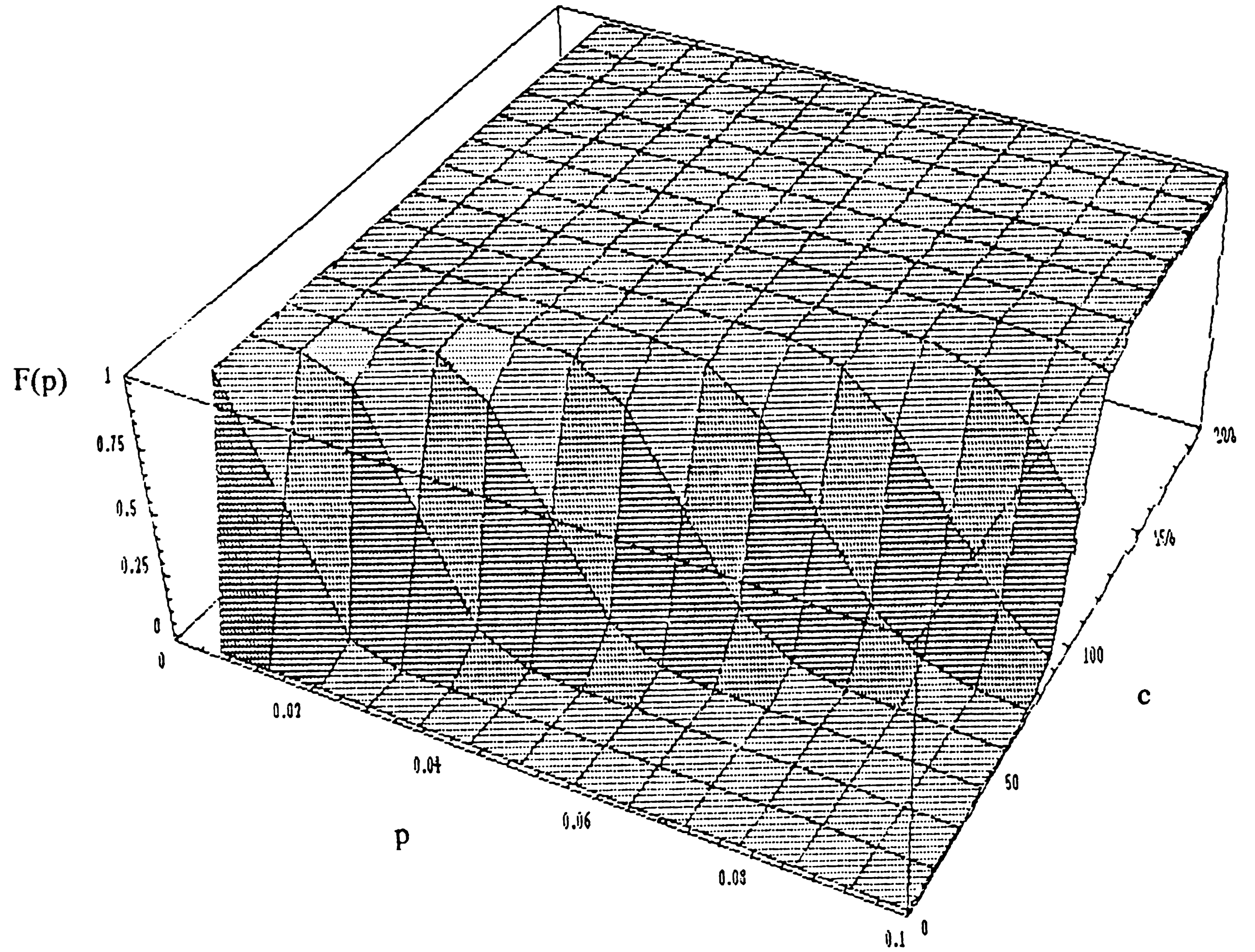

Figure A.4. F(p) as a Function of p and c, when n = 1000